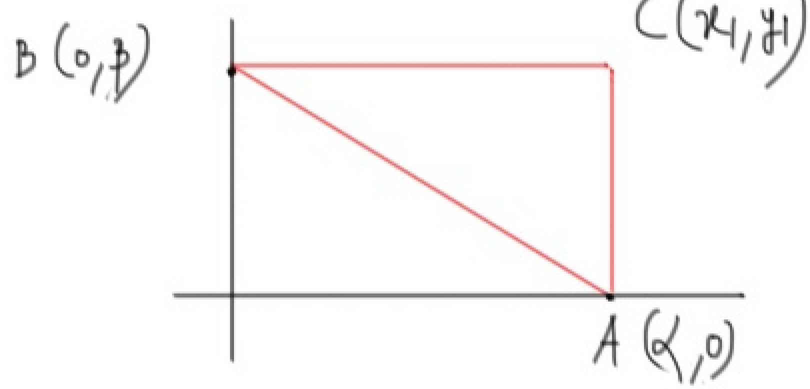
In Δ ABC, $m\angle C = 90^\circ$, $l(CA) = a$ & $l(CB) = b$. The triangle is placed in the XY plane in such a way that the angular points A & B slide along the X & Y axes respectively. The locus of point C is

A) ellipse
B) circle
C) parabola
D) pair of straight lines.



$$AC^2 = a^2 = (x_1-\alpha)^2 + y_1^2 \quad \&$$

$$BC^2 = b^2 = x_1^2 + (y_1-\beta)^2.$$

$$\& \quad AB^2 = \alpha^2+\beta^2 = a^2+b^2.$$

$$\therefore \alpha = x_1 - \sqrt{a^2-y_1^2} \quad \& \quad \beta = y_1 - \sqrt{b^2-x_1^2}.$$

$$\& \left(x_1 - \sqrt{a^2-y_1^2}\right)^2 + \left(y_1 - \sqrt{b^2-x_1^2}\right)^2 = a^2+b^2$$

$$\therefore \cancel{x_1^2} - 2x_1\sqrt{a^2-y_1^2} + \cancel{a^2} - \cancel{y_1^2} + \cancel{y_1^2} - 2y_1\sqrt{b^2-x_1^2} + \cancel{b^2} - \cancel{x_1^2} = \cancel{a^2} + \cancel{b^2}$$

$$\therefore \quad 4x_1^2(a^2-y_1^2) = 4y_1^2(b^2-x_1^2)$$

$$\therefore x_1^2a^2 - x_1y_1^2 = y_1^2b^2 - x_1y_1^2$$

$$\therefore x_1^2a^2 - y_1^2b^2 = 0. \quad \Big\} \text{ pair of straight lines.}$$

The angle between asymptotes of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

A) $2\tan^{-1}(e)$
B) $\tan^{-1}(1/e)$
C) $\tan^{-1}(2/e)$
D) $2\sec^{-1}(e)$

Let $a = b = 1$. $\therefore$ we get a rectangular hyperbola $(e = \sqrt{2})$.

This hyperbola has asymptotes $x^2 - y^2 = 0$ or $x = y$ & $x = -y$.

(Equation of asymptotes always differs in constant term, when compared with equation of conic.)

Here, angle $= 90^\circ = 2 \times 45^\circ = 2 \times \sec^{-1}(\sqrt{2}) = 2 \times \sec^{-1}(e)$ (D)

Angle between two planes passing through the line $x+y=1$ & $z=0$, making an angle of $\sin^{-1}\left(\frac{1}{3}\right)$ with $x+y+z=0$ is

A) $\cos^{-1}\left(\frac{1}{3}\right)$ C) $\sec^{-1}\left(\frac{9}{7}\right)$

B) $\sin^{-1}\left(\frac{1}{3}\right)$ D) $\tan^{-1}\left(\frac{7}{9}\right)$.

The general equation of plane through given line is $x+y+\lambda z-1=0$.

D.r.s of normal to this plane : $1, 1, \lambda$.

$\therefore$ we've $\cos(\theta) = \left|\dfrac{1+1+\lambda}{\sqrt{3}\ \sqrt{2+\lambda^2}}\right| = \left|\dfrac{\lambda+2}{\sqrt{3(\lambda^2+2)}}\right|$

$\therefore \sin^2(\theta) = 1 - \dfrac{(\lambda+2)^2}{3(\lambda^2+2)} = \dfrac{1}{9}$, given that $\theta = \sin^{-1}\left(\frac{1}{3}\right)$.

On solving the quadratic, $\lambda = \frac{2}{5}$ or $\lambda = 2$.

$\therefore$ the required planes are $x+y+\frac{2}{5}z-1=0$ or $x+y+2z-1=0$.

Angle betn them $= \cos^{-1}\left|\dfrac{1+1+\frac{4}{5}}{\sqrt{2+\frac{4}{25}}\ \sqrt{6}}\right| = \cos^{-1}\left|\dfrac{14}{\sqrt{54(6)}}\right| = \cos^{-1}\left(\frac{14}{18} = \frac{7}{9}\right)$ C

$\lim\limits_{x \to c} f(x) + g(x) = 3$ & $\lim\limits_{x \to c} f(x) - g(x) = -1$.

Then, $\lim\limits_{x \to c} f(x) \times g(x) =$

A) 1 B) 2 C) 3 D) 4

Let $\lim_{x \to c} f(x) = 1$ & $\lim_{x \to c} g(x) = 2$.

$\therefore$ both of the given conditions are satisfied.

$\therefore$ required limit $= 1 \times 2 = 2$.

The alternate method involves $(a+b)^2 - (a-b)^2 = 4ab$.

Consider $f(x) = x^2 \times \sin\left(\frac{1}{x}\right)$ & $g(x) = x \times \sin\left(\frac{1}{x}\right)$, $x \neq 0$.
$f(0) = g(0) = 0$.

A) Both the curves $y = f(x)$ & $g(x)$ have tangents at origin.
B) None of them has tangent at origin.
C) $y = f(x)$ has, but $y = g(x)$ doesn't.
D) $y = g(x)$ has, but $y = f(x)$ doesn't.

Observe: By sandwich principle, both functions are continuous at $x=0$.

$$f'(0) = \lim_{h\to 0} \frac{h^2 \times \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \times \sin\left(\frac{1}{h}\right) = 0.$$

$$\left.\begin{aligned} g'(0) = \lim_{h\to 0} \frac{h \times \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \end{aligned}\right\} \text{doesn't exist.}$$

(C)

Consider $F(x) = (x-a)^2 \times (x-b)^2 + x$. There always exists a number x in (a,b) s.t. $F(x) = \frac{a+b}{2}$. This statement is

A) True, when $a < 0$, $b < 0$.

B) True, when $a > 0$, $b > 0$.

C) True, when both a & b are of opposite signs.

D) Always true.

$F(x) = (x-a)^2 \times (x-b)^2 + x$ is continuous everywhere.

$F(a) = a$ & $F(b) = b$.

$\therefore$ by intermediate value theorem, $\exists x \in (a,b)$, s.t. $F(x) = \frac{a+b}{2}$,

as $a < \frac{a+b}{2} < b$.

$$\int_{2}^{3} \frac{1}{x^7 - x}\, dx =$$

A) $\ln\left(\frac{1}{126}\right)$

B) $\frac{1}{6}\ln\left(\frac{1}{126}\right)$

C) $\frac{1}{6}\ln\left(\frac{6656}{6561}\right)$

D) $\frac{1}{6}\ln\left(\frac{7744}{7533}\right)$

Observe that $\forall x \in [2,3]$, $\frac{1}{x^7-x} > 0$. $\therefore \int_2^3 \frac{1}{x^7-x}dx > 0$.

$\therefore$ (A) & (B) gone.

Presence of $\frac{1}{6}$ & $\ln$ suggests that a substitution involving $\frac{1}{x^6}$ might work.

$$\int \frac{1}{x^7-x}dx = \int \frac{1}{x^7\left(1-\frac{1}{x^6}\right)}dx = \int \frac{1}{t} \times \frac{1}{6}\,dt = \frac{1}{6}\ln(t) + C.$$

$$t = 1-\frac{1}{x^6}, \quad \frac{dt}{6} = \frac{dx}{x^7}.$$

Putting the limits, $\frac{1}{6}\left[\ln\left(1-\frac{1}{x^6}\right)\right]_2^3 = \frac{1}{6}\ln\left[\left(1-\frac{1}{3^6}\right) \Big/ \left(1-\frac{1}{2^6}\right)\right]$

$$= \frac{1}{6}\ln\frac{6656}{6561} \quad \text{(C)}$$

Suppose that $f(x)$ is a non-constant positive function with continuous first derivative. Then

$$\lim_{n \to \infty} \int_0^1 f(x) \sin(nx)\, dx =$$

A) 0 B) $-\infty$ C) ∞ D) $\frac{1}{3}$.

Let $f(x)$ be $x+1$. satisfies both requirements in the problem.

$$\therefore \int_0^1 f(x)\sin(nx)\,dx = \int_0^1 (x+1)\sin(nx)\,dx = \left[(x+1)\times\frac{-\cos(nx)}{n}\right]_0^1 - \int_0^1 \frac{-\cos(nx)}{n}\,dx$$

by parts.

$$= \frac{-2\cos(n)+1}{n} + \frac{\sin(n)}{n^2}$$

As $\sin(n)$ & $\cos(n)$ are bounded, $\lim_{n\to\infty}\int_0^1 f(x)\sin(nx)\,dx = 0.$ (A)

If $x \sin(\pi x) = \int_0^{x^2} f(t)\,dt$, where f is a continuous function, $f(4) =$

A) $\frac{\pi}{4}$ B) $\frac{\pi}{2}$ C) $\frac{3\pi}{4}$ D) π.

Differentiating w.r.t. (x),

$1 \times \sin(\pi x) + x \times \cos(\pi x) \times \pi = 0 + f(x^2) \times 2x - 0$, by Leibniz rule.

(known as Differentiation under Integral Sign, later!)

Letting $x = 2$,

$$0 + 2\pi \times 1 = f(4) \times 4.$$

$$\therefore f(4) = \pi/2. \quad (B)$$

Suppose $B = \begin{bmatrix} c-2a & 0 \\ 2b-c & a-2b \end{bmatrix}$. Given that $B^4 = \begin{bmatrix} 16 & 0 \\ -20 & 1 \end{bmatrix}$ & a, b, c are integers, $a+b+c$ can be

A only 13
B) only -13
C) ± 13
D) any real number.

Let $c-2a=x$, $2b-c=y$ & $a-2b=z$.

$$\therefore B=\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \text{ \& } B^4=\begin{bmatrix} x^4 & 0 \\ y(x+z)(x^2+z^2) & z^4 \end{bmatrix}=\begin{bmatrix} 16 & 0 \\ -20 & 1 \end{bmatrix}$$

$\therefore x^4=16$, $z^4=1$ $\quad \therefore x=\pm 2$, $z=\pm 1$.

If $x=2$, $z=1$, $y=\dfrac{-20}{3(5)}$, not possible. Similarly $x=-2$ & $z=-1$ is ruled out. $(y \in \mathbb{Z})$.

$\therefore$ if $x=2$ & $z=-1$, we get $y=\dfrac{-20}{5}=-4$ &

if $x=-2$ & $z=1$, we get $y=\dfrac{-20}{-5}=4$.

$$\therefore \left.\begin{aligned} c-2a &= x \\ 2b-c &= y \\ a-2b &= z \end{aligned}\right\} \text{ On solving simultaneously, we get } 3, 2, 8 \text{ \& } -3, -2, -8$$

$\therefore a+b+c=\pm 13$.

Suppose S is a set consisting of 6 elements. The number of ways of selecting 2 not necessarily distinct subsets of S, s.t. their union is S, is ______. Assume that the pair of chosen subsets is not an ordered pair, while counting.

A) 1008

B) 1007

C) 364

D) 365

Let $x \in S$. For $A \cup B = S$,

$x \in A, x \notin B$ or $x \notin A, x \notin B$ or $x \in A, x \in B$.

$n(S) = 6$.

$\therefore$ there are 3^6 ways to choose A & B.

Except for pairs $A = B$, 3^6 contains each pair twice.

As $A \cup B = S$ with $A = B$ occurs iff $A = B = S$,

total number of ways $= \dfrac{3^6 - 1}{2} + 1 = 365$.

Let $f(x) = e^{-x}$, $x \geq 0$ be the pdf of a continuous random variable X. Then, the median of X is

A) e B) $\frac{1}{e}$ C) $\ln(2)$ D) none of these.

Let median be at $x = \alpha$. $\therefore \displaystyle\int_0^\alpha e^{-x}\,dx = \int_\alpha^\infty e^{-x}\,dx = \frac{1}{2}$.

$$\therefore \left[\frac{e^{-x}}{-1}\right]_\alpha^\infty = \frac{1}{2}.$$

$$\frac{0 - e^{-\alpha}}{-1} = \frac{1}{2}$$

$$\therefore e^{-\alpha} = \frac{1}{2}$$

$$\therefore -\alpha = -\ln(2).$$

$$\therefore \alpha = \ln(2)$$

Let $f(x) = \sin(x^3)$. Then $\left(\frac{d^{15}f}{dx^{15}}\right)_{x=0}$ is

A) Zero
B) 10
C) 15
D) none of these

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\therefore \sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

Also,

$$f(x) = f(0) + x \times f'(0) + \frac{x^2}{2!} \times f''(0) + \cdots + \frac{x^{15}}{15!} \times f^{15}(0) + \cdots$$

$$\therefore \frac{f^{15}(0)}{15!} = \frac{1}{5!}$$

$$\therefore f^{15}(0) = \frac{15!}{5!}$$ (D)

On eliminating (θ) from the equations ① & ②, we get equation ③.
Then, $\alpha+\beta+\gamma =$

$$\frac{x}{a}\cos(\theta) - \frac{y}{b}\sin(\theta) = \cos(2\theta), \qquad ①$$

$$\frac{x}{a}\sin(\theta) + \frac{y}{b}\cos(\theta) = 2\sin(2\theta), \qquad ②$$

$$\left(\frac{x}{a}+\frac{y}{b}\right)^{\alpha} + \left(\frac{x}{a}-\frac{y}{b}\right)^{\beta} = \gamma. \qquad ③$$

① $8/3$

② $7/3$

③ 3

④ $10/3$

Multiplying ① by $\cos(\theta)$, ② by $\sin(\theta)$ & adding, $\frac{x}{a} = \cos(\theta)\cos(2\theta) + 2\sin(\theta)\sin(2\theta)$

② by $\cos(\theta)$, ① by $\sin(\theta)$ & subtracting, $\frac{y}{b} = 2\sin(2\theta)\cos(\theta) - \cos(2\theta)\sin(\theta)$.

On simplifying by using double angle formulas,

$$\frac{x}{a} + \frac{y}{b} = \left[\sin(\theta) + \cos(\theta)\right]^3 \quad \& \quad \frac{x}{a} - \frac{y}{b} = \left[\cos(\theta) - \sin(\theta)\right]^3$$

Do this once on your own!

$$\therefore \sin(\theta) + \cos(\theta) = \left(\frac{x}{a} + \frac{y}{b}\right)^{1/3} \quad \& \quad \cos(\theta) - \sin(\theta) = \left(\frac{x}{a} - \frac{y}{b}\right)^{1/3}$$

Squaring & adding,

$$2 = \left(\frac{x}{a} + \frac{y}{b}\right)^{2/3} + \left(\frac{x}{a} - \frac{y}{b}\right)^{2/3}$$

$$\therefore \alpha + \beta + \gamma = \frac{2}{3} + \frac{2}{3} + 2 = 10/3$$

(D)

In ΔABC, $\tan\left(\frac{A}{2}\right) = \frac{5}{6}$ & $\tan\left(\frac{B}{2}\right) = \frac{20}{37}$. Then

A) a, b, c are in AP

B) a, c, b are in AP.

C) a, b, c are in GP

D) a, c, b are in GP.

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right) = \cot\left(\frac{C}{2}\right) = \frac{\frac{5}{6}+\frac{20}{37}}{1-\frac{5}{6}\times\frac{20}{37}} = \frac{5}{2} \Rightarrow \tan\left(\frac{C}{2}\right) = \frac{2}{5}$$

$$\sin(A) = \frac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} = \frac{60}{61}, \quad \sin(B) = \frac{1480}{1769} \ \& \ \sin(C) = \frac{20}{29}$$

$$a = \frac{60\lambda}{61}, \quad b = \frac{1480\lambda}{1769}, \quad c = \frac{20\lambda}{29}$$ } by sine rule

$$\frac{a+c}{2} = \lambda\left[\frac{30}{61}+\frac{10}{29}\right] = \frac{1480\lambda}{1769} = b.$$ (A)